MTH 201: Multivariable Calculus and Differential Equations

Semester 1, 2014-15

1. Three-dimensional geometry

1.1. Lines and planes.

(i) Vector, parametric, and symmetric equation of a line.
(ii) Vector and scalar equations of a plane.

1.2. Cylinders and quadric surfaces.

(i) Cylinders with examples.
(ii) General equation of a quadric surface with examples.

2. Multivarable Differential Calculus

2.1. Scalar and vector fields.

(i) Scalar and vector fields.
(ii) Open balls and open sets.
(iii) Interior, exterior, and boundary of a set.
(iv) Theorem: If A_1 and A_2 are open sets in $\mathbb{R}^1$, then $A_1 \times A_2$ is open in $\mathbb{R}^2$.

2.2. Limits and continuity.

(i) Limits and continuity.
(ii) Theorem: If $\lim_{x \to a} f(x) = b$ and $\lim_{x \to a} g(x) = c$, then:
  (a) $\lim_{x \to a} (f(x) + g(x)) = b + c$
  (b) $\lim_{x \to a} \lambda f(x) = \lambda b$, for every scalar λ
  (c) $\lim_{x \to a} f(x)g(x) = bc$
  (d) $\lim_{x \to a} \|f(x)\| = \|b\|$
(iii) Components of a vector field.
(iv) Theorem: A vector field is continuous if and only if each of its components are continuous.
(v) The identity function and polynomial functions are continuous on $\mathbb{R}^n$.
(vi) The composition of two continuous functions is continuous.

(vii) Example of a discontinuous scalar field that is continuous in each variable.

(viii) Derivative $f'(a; v)$ of a scalar field f with respect to a vector v.

(ix) Mean Value Theorem (for scalar fields) : Let $f'(a + tv, v)$ exist for each $t \in [0, 1]$. Then for some real number $\theta \in (0, 1)$, we have

$$f(a + y) - f(a) = f'(z; y), \text{ where } z = a + \theta v.$$

2.3. **Directional, partial and total derivatives.**

(i) Directional derivatives and partial derivatives.

(ii) Directional derivatives and continuity.

(iii) Differentiable scalar field.

(iv) The total derivative.

(v) Theorem: If f is differentiable at a with total derivative T_a, then the derivative $f'(a; v)$ exists for every $a \in \mathbb{R}^n$, and we have

$$T_a(v) = f'(a; v).$$

Moreover, if $v = (v_1, \ldots, v_n)$, we have

$$f'(a; v) = \sum_{k=1}^{n} D_k f(a) v_k.$$

2.4. **Gradient and tangent planes.**

(i) The gradient of a scalar field.

(ii) Theorem: If a scalar field is differentiable at a, then f is contiunuous at a.

(iii) Sufficient condition for differentiability: If f has partial derivatives $D_1 f, \ldots, D_n f$ in some n-ball $B(a)$ and are continuous at a, then f is differentiable at a.

(iv) Theorem (Chain rule for scalar fields): Let f be a scalar field defined on an open set S in $\mathbb{R}^n$, and let r be a vector-valued function which maps an interval J from $\mathbb{R}^1$ into S. Define the composite $g = f \circ r$ on J by the $g(t) = f(r(t))$, if $t \in J$. Let $t \in J$ such that $r'(t)$ exists and assume that f is differentiable at $r(t)$.Then $g'(t)$ exists and is given by

$$g'(t) = \nabla f(a) \cdot r'(t),$$

where $a = r(t)$.

(v) Level sets and tangent planes.

2.5. **Derivatives of vector fields.**

(i) Derivatives of vector fields.

(ii) Theorem: If f is differentiable at a with total derivative T_a, then the derivative $f'(a;v)$ exists for every $a \in \mathbb{R}^n$, and we have

$$T_a(v) = f'(a;v).$$

Moreover, if $f = (f_1, \ldots, f_m)$ and $v = (v_1, \ldots, v_n)$, we have

$$T_a(v) = \sum_{k=1}^{m} \nabla f_k(a) \cdot e_k = (\nabla f_1(a) \cdot v, \ldots, \nabla f_m(a) \cdot v).$$

(iii) Theorem: If a vector field is differentiable at a, then f is continuous at a.

(iv) Theorem (Chain Rule): Let f and g be vector fields such that the composition $h = f \circ g$ is defined in a neighborhood of a point a. Assume that g is differentiable at a, with total derivative $g'(a)$. Let $b = g(a)$ and assume that f is differentiable at b, with total derivative $f'(b)$. Then h is fifferentiable at a, and the total derivative $h'(a)$ is given by

$$h'(a) = f'(b) \circ g'(a).$$

(v) Matrix form of chain rule.

(vi) Theorem (A sufficient condition for equality of mixed partial derivatives): Assume f is a scalar field such that the partial derivatives $D_1 f$, $D_2 f$, $D_{1,2} f$, and $D_{2,1} f$ exist on an open set S. If $f(a,b)$ is a point in S at which both $D_{1,2} f$ and $D_{2,1} f$ are continuous, we have

$$D_{1,2} f(a,b) = D_{2,1} f(a,b).$$

(vii) Theorem (Sufficient condition for equality of mixed partial derivatives): Assume f is a scalar field such that the partial derivatives $D_1 f$, $D_2 f$, and $D_{2,1} f$ exist on an open set S containing. Assume further that $D_{2,1} f$ if continuous on S. Then the derivative $D_{1,2} f(a,b)$ exists and we have

$$D_{1,2} f(a,b) = D_{2,1} f(a,b).$$

3. Applications of derivatives

3.1. Partial differential equations.

(i) First order PDE of the form

$$a\frac{\partial f(x,y)}{\partial x}\partial x + b\frac{\partial f(x,y)}{\partial y} = 0.$$

(ii) Theorem: Let g be differentiable on $\mathbb{R}^1$, and let f be the scalar field defined on $\mathbb{R}^2$ by the equation

$$f(x,y) = g(ax - by),$$

where a and b are constants, not both zero. Then f satisfies the first-order PDE

$$a\frac{\partial f(x,y)}{\partial x}\partial x + b\frac{\partial f(x,y)}{\partial y} = 0 \tag{1}$$

everywhere in $\mathbb{R}^2$. Conversely, every differentiable solution necessarily has the form $g(x,y) = f(ax - by)$.

3.2. The one-dimensional wave equation.

(i) the one-dimensional wave equation

$$\frac{\partial^2 f}{\partial t^2} = c^2\frac{\partial^2 f}{\partial x^2}.$$

(ii) Theorem (D'Alembert's solution to the wave equation): Let F and G be given by functions such that G is differentiable and F is twice differentiable on $\mathbb{R}^1$. Then the function f is given by the formula

$$f(x,t) = \frac{F(x+ct) + F(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)ds$$

satisfies the wave equation

$$\frac{\partial^2 f}{\partial t^2} = c^2\frac{\partial^2 f}{\partial x^2}$$

and initial conditions $f(x,0) = F(x)$, $D_2 f(x,0) = G(x)$. Conversely, any function with equal mixed partials which satisfies the wave equation necessarily has the above form.

3.3. **The derivative of functions defined implicitly.**

(i) Theorem: Let F be a scalar field diferentiable on an open set T in $\mathbb{R}^n$. Assume that the equation

$$F(x_1, \ldots, x_n) = 0$$

defines x_n explicitly as a differentiable function of $x_1, \ldots, x_{n-1}$, say

$$x_n = f(x_1, \ldots, x_{n-1}),$$

for all $(x_1, \ldots, x_{n-1})$ in an open set $S \subset \mathbb{R}^{n-1}$. Then for each $k = 1, 2, \ldots, n-1$, the partial derivative $D_k f$ is given by the formula

$$D_k f = -\frac{D_k F}{D_n F}$$

for all points at which $D_n F \neq 0$. The partial derivatives $D_k f$ and $D_n f$ are to be evaluated at $(x_1, x_2, \ldots, x_{n-1}, f(x_1, \ldots, x_{n-1}))$.

(ii) Jacobian determinant notation $\dfrac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}$.

(iii) Suppose that we have two surfaces with implicit representations $F(x, y, z) = 0$ and $G(x, y, z) = 0$ that intersect along a curve C. Suppose that it is possible to solve x and y in terms of z, and the solutions are given by the equations $x = X(z)$ and $y = Y(z)$. Then

$$X'(z) = \frac{\partial(F, G)/\partial(y, z)}{\partial(F, G)/\partial(x, y)}, \; Y'(z) = \frac{\partial(F, G)/\partial(z, x)}{\partial(F, G)/\partial(x, y)}.$$

3.4. **Maximum and minimum values.**

(i) Local maximum and minimum values of a function.

(ii) Theorem: A scalar field $f(x, y)$ has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = f_y(a, b) = 0$.

(iii) Critical (or stationary) points and saddle points of a function.

(iv) Theorem (Second-derivative test): Let (a, b) be a critical point of a scalar field $f(x, y)$ with continuous second-order partial derivatives in a 2-ball (or disk) with center (a, b). Let $A = f_{xx}(a, b)$, $B =$

$f_{xy}(a,b)$, $C = f_{yy}(a,b)$, and let

$$D = \det \begin{bmatrix} A & B \\ B & C \end{bmatrix} = AC - B^2.$$

Then we have:

(a) If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.

(c) If $D < 0$, then $f(a,b)$ is neither a local maximum nor a local minimum (or (a,b) is a saddle point).

(d) If $D = 0$, then the test is inconclusive.

(v) Theorem (Extreme value theorem): If a scalar field $f(x,y)$ is continuous on a closed and bounded (i.e. compact) set $A \in \mathbb{R}^2$, then f attains its absolute maximum and absolute minimum values at points in A.

(vi) Absolute maximum and minimum values.

(vii) Theorem (Lagrange's Theorem): Let f and g have continuous partial derivatives such that f has an extremum at a point (a,b) on the smooth constraint curve $g(x,y) = c$. If $\nabla g(a,b) \neq 0$, there exists a real number λ such that

$$\nabla f(a,b) = \lambda \nabla g(a,b).$$

(viii) Method of Lagrange multipliers: Let f and g satisfy the hypothesis of Lagrange's Theorem, and let f have a maximum or minimum subject to the constraint $g(x,y) = k$. To find the extremal values of f, we use the following steps:

Step 1: Simultaneously solve the equations $\nabla f(x,y) = \lambda g(x,y)$ and $g(x,y) = k$ by solving the following system of equations.

$$f_x(x,y) = \lambda g_x(x,y)$$
$$f_y(x,y) = \lambda g_y(x,y)$$
$$g(x,y) = k$$

Step 2: Evaluate f at each solution point obtained in Step 1. The largest value yields the maximum of f subject to the constraint, and the smallest value yields the minimum of f subject to the constraint.

4. MULTIVARIABLE INTEGRAL CALCULUS

4.1. **Double Integrals.**

(i) Intepreting the double inegral as volume of the solid bounded below by a region R and above by $z = f(x, y)$.

(ii) Double integrals over a rectangular region $R : [a, b] \times [c, d]$.

(iii) Thorem (Fubini weaker form): If $f(x, y)$ is continuous throughout the region $R : [a, b] \times [c, d]$, then

$$\iint_R f(x,y)dA = \int_a^b\int_c^d f(x,y)dxdy = \int_a^b\int_c^d f(x,y)dydx.$$

(iv) Double integrals over a general region R.

(v) Thorem (Fubini stronger form): Let $f(x, y)$ be continuous on a region R.

(a) If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x,y)dA = \int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)dydx.$$

(b) If R is defined by $c \le y \le d$, $h_1(x) \le x \le h_2(x)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x,y)dA = \int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)dxdy.$$

(vi) If $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R, then the following properties hold.

(a) $\iint_R cf(x,y)dA = c\iint_R f(x,y)dA$, for any $c \in \mathbb{R}$.

(b) $\iint_R (f(x,y) \pm g(x,y))dA = \iint_R f(x,y)dA \pm \iint_R g(x,y)dA.$

(c) $\iint_R f(x,y)dA \ge 0$, if $f(x, y) \ge 0$ on R.

(d) $\iint_R f(x,y)dA \ge \iint_R g(x,y)dA$, if $f(x, y) \ge g(x, y)$ on R.

(e) $\iint_R f(x,y)dA \ge \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA$, if R is the union of two nonoverlapping regions R_1 and R_2.

(vii) The *area* of a closed, bounded plane region R is given by

$$A = \iint_A dA.$$

(viii) The average value of an integrable function f over a region R of area A is given by

$$\text{The average value of } f \text{ over } R = \frac{1}{A}\iint_R f\, dA.$$

4.2. **Triple integrals.**

(i) Integrability of $F(x,y,z)$ over a closed, bounded region D in $\mathbb{R}^3$.

(ii) The *volume* of a closed, bounded region D in space is given by

$$V = \iiint_D dV.$$

4.3. **Substitution in multiple integrals.**

(i) Theorem (Substitution in double integrals): Suppose that a region G in the uv-plane is transformed one-to-one into a region R in the xy-plane by equations of the form $x = g(u,v)$ and $y = h(u,v)$. Then any function $f(x,y)$ defined on R can be thought of as a function $f(g(u,v), h(u,v))$ defined on G. Moreover, if g, h, and f have continuous partial derivatives and $\frac{\partial(x,y)}{\partial(u,v)}$ is zero only at isolated points, then

$$\iint_R f(x,y)\, dx\, dy = \iint_G f(g(u,v), h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv.$$

(ii) Theorem (Substitution in triple integrals): Suppose that a region G in the uvw-space is transformed one-to-one into a region R in the xyz-space by equations of the form $x = g(u,v,w)$, $y = h(u,v,w)$, and $z = k(u,v,w)$. Then any function $f(x,y,z)$ defined on R can be thought of as a function $f(g(u,v,w), h(u,v,w), k(u,v,w))$ defined on G. Moreover, if g, h, and k have continuous first partial derivatives and $\frac{\partial(x,y)}{\partial(u,v)}$ is zero only at isolated points, then

$$\iiint_D f(x,y)\, dx\, dy\, dz = \iiint_G f(g(u,v,w), h(u,v,w), k(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\, dv\, dw.$$

4.4. **Double integrals in polar coordinates.**

(i) *Polar Coordinates* represent a point P by the ordered pairs (r, θ) in which
  (a) r is distance of P from ther origin O, and
  (b) θ is the directed angle from the initial ray originating at O (along the positive direction of x-axis) to the ray OP.

(ii) The rectangular (x, y) and polar (r, θ) coordinate systems are related by the following set of equations

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ r^2 = x^2 + y^2,\ \text{and}\ \tan\theta = \frac{y}{x}.$$

(iii) Theorem: The area of a closed and bounded region R in polar coordinates is given by

$$A = \iint_R r\,dr\,d\theta.$$

(iv) Theorem (Double integral in polar coordinates): Suppose that a region G in the $r\theta$-plane is transformed one-to-one into a region R in the xy-plane by polar equations of the form $x = r\cos\theta$ and $y = r = \sin\theta$. Then

$$\iint_R f(x, y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta) r\,dr\,d\theta.$$

4.5. **Triple integrals in cylindrical and spherical cordinates.**

(i) *Cylindrical coordinates* represent a point P in space ordered by triples (r, θ, z) in which
  (a) r and θ are the polar coordinates of the verical projection of P onto the xy-plane, and
  (b) z is the rectangular vertical coordinate.

(ii) The equations relating rectangular (x, y, z) and cylindrical (r, θ, z) coordinates are:

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ z = z,$$
$$r^2 = x^2 + y^2,\ \tan\theta = \frac{y}{x}.$$

Theorem (Triple integral in cylindrical coordinates): Suppose that a region G in the $r\theta z$-space is transformed one-to-one into a region

R in the xyz-space by cylindrical equations of the form $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$. Then

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_G f(r\cos\theta, r\sin\theta, z) r\,dz\,dr\,d\theta.$$

(iii) *Spherical coordinates* represent a point P in space ordered by triples (ρ, ϕ, θ) in which
  (a) ρ is the distance from P to the origin,
  (b) θ is the angle from cylindrical coordinates $(0 \le \theta \le 2\pi)$, and
  (c) ϕ is the angle $\overline{OP}$ makes with the positive z-axis $(0 \le \phi \le \pi)$.

(iv) The equations relating spherical (ρ, θ, ϕ) to cartesian (x, y, z) and cylindrical (r, θ, z) coordinates are:

$$r = \rho\sin\phi,\ x = r\cos\theta = \rho\sin\phi\cos\theta,$$
$$z = \rho\cos\phi,\ y = r\sin\theta = \rho\sin\phi\sin\theta,$$
$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2}.$$

(v) Theorem (Triple integral in spherical coordinates): Suppose that a region G in the $\rho\phi\theta$-space is transformed one-to-one into a region R in the xyz-space by spherical equations of the form $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$. Then

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_G f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

4.6. **Line integrals.**

(i) *Line Integral:* Let F be a vector field with continuous components defined along a smooth curve C parametrized by $r(t)$, $t \in [a,b]$. Then the *line integral of F along C* is

$$\int_C F\cdot T\,ds = \int_C \left(F\cdot\frac{dr}{ds}\right)\,ds = \int_C F\cdot dr = \int_a^b F(r(t))\cdot\frac{dr}{dt}\,dt.$$

$$r = \rho\sin\phi,\ x = r\cos\theta = \rho\sin\phi\cos\theta,$$
$$z = \rho\cos\phi,\ y = r\sin\theta = \rho\sin\phi\sin\theta,$$
$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2}.$$

(ii) Let C be a smooth curve parametrized by $r(t)$, $t \in [a, b]$, and F be a continuous force field over a region containing C. Then the *work done* in moving an object from the point $A = r(a)$ ro the point $B = r(b)$ along C is given by

$$W = \int_C F \cdot T\, ds = \int_a^b F(r(t)) \cdot \frac{dr}{dt}\, dt.$$

(iii) Let $F = Mi + Nj + Pk$ defined along the smooth curve $C : r(t) = g(t)i + h(t)j + k(t)k$, $t \in [a, b]$. Then the equivalent forms of the work integral are:

$$\begin{aligned} W &= \int_C F \cdot T\, ds \\ &= \int_c F \cdot dr \\ &= \int_a^b F \cdot \frac{dr}{dt} dt \\ &= \int_a^b \left(M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt} \right) dt \\ &= \int_C M\, dx + N\, dy + P\, dz. \end{aligned}$$

(iv) If $r(t)$ parametrizes a smooth curve C in the domain of a continuous velocity field F, then the *flow* along the curve from $A = r(a)$ to $B = r(b)$ is given by

$$\int_C F \cdot T\, ds.$$

(v) If C is a smooth simple closed curve in the domain of a continuous vector field $F = M(x, y)i + N(x, y)j$ in the plane, and if N is the outward pointing unit normal vector on C, the *flux* of F across C is given by

$$\int_C F \cdot n\, ds.$$

Furthermore, if $F = Mi + Nj$, then the flux of F across C is given by

$$\int_C M\, dy - N\, dx.$$

4.7. **Green's Theorem.**

(i) The *divergence (or the flux density)* of a vector field $F = Mi + Nj$ at the point (x, y) is

$$\operatorname{div} F = \nabla \cdot F = \frac{\partial M}{\partial x} + \frac{\partial M}{\partial y}.$$

(ii) The *circulation density* of a vector field $F = Mi + Nj$ at the point (x, y) is the scalar expression

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

This expression is also called the *k-component of the curl*, denoted by $(\operatorname{curl} F) \cdot k$ or $(\nabla \times F) \cdot k$.

(iii) Green's Theorem (Flux-divergence or Normal Form): Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $F = Mi + Nj$ be a vector field with M and N having continuous first partial derivatives in a open region containing R. Then the outward flux of F across C equals the double integral of $\operatorname{div} F = \nabla \cdot F$ over the region R enclosed by C, that is,

$$\int_C F \cdot n\, ds = \int_C M\, dy - N\, dx = \iint_R (\nabla \cdot F)\, dx\, dy.$$

(iv) Green's Theorem (Circulation-Curl or Tangential Form): Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $F = Mi + Nj$ be a vector field with M and N having continuous first partial derivatives in a open region containing R. Then the counterclockwise circulation of F around C equals the double integral of $(\operatorname{curl} F) \cdot k = (\nabla \times F) \cdot k$ over R, that is,

$$\int_C F \cdot T\, ds = \int_C M\, dx + N\, dy = \iint_R ((\nabla \times F) \cdot k)\, dx\, dy.$$

4.8. **Surface Integrals.**

(i) The area of the smooth surface

$$r(u, v) = f(u, v)i + g(u, v)j + h(u, v)k, (u, v) \in [a, b] \times [c, d]$$

is given by

$$A = \iint_R |r_u \times r_v|\, dA = \int_c^d \int_a^b |r_u \times r_v|\, du\, dv.$$

(ii) The area of the surface $f(x, y, z) = c$ over a closed and bounded plane region R is given by

$$\int_R \frac{|\nabla F|}{|\nabla F \cdot p|}\, dA,$$

where P is $p = i, j$, or k is the unit normal vector to R and $\nabla F \cdot p \neq 0$.

(iii) For a smooth surface defined parametrically as

$$r(u, v) = f(u, v)i + g(u, v)j + h(u, v)k, (u, v) \in R$$

and a contiunuous function $G(x, y, z)$ defined on S, the surface integral of G over S is given by

$$\iint_S g(x, y, z)\, d\sigma = \iint_R G(f(u, v), g(u, v), h(u, v))|r_u \times r_v|\, du\, dv.$$

(iv) For a surface S given implicitly by $f(x, y, z) = c$, where F is a continuously differentiable function, with S lying above its closed and bounded shadow region R in the coordinate plane beneath it, the surface integral of the continuous function G over S is given by

$$\iint_S g(x, y, z)\, d\sigma = \iint_R g(x, y, z) \frac{|\nabla F|}{|\nabla F \cdot p|}\, dA,$$

where P is $p = i, j$, or k is the unit normal vector to R and $\nabla F \cdot p \neq 0$.

(v) For a surface S given explicitly as a graph $z = f(x, y)$, where f is a continuously differentiable function over a regions R in the xy-plane, the surface integral of the continuous function G over S is given by

$$\iint_S g(x, y, z)\, d\sigma = \iint_R G(x, y, f(x, y))\sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy.$$

4.9. **Stokes' Theorem.**

(i) Stokes' Theorem: Let S be a piecewise smooth oriented surface having a piecewise smooth boundary curve C. Let $F = Mi + Nj + Pk$ be a vector field whose components have continuous first partial derivatives on an open region containing S. Then the circulation of F around C in the direction counterclockwise with respect to the surface's unit normal vector n equals the integral of $(\nabla \times F) \cdot n$ over S, that is,

$$\int_C F \cdot dr = \iint_S (\nabla \times F) \cdot n \, d\sigma.$$

4.10. **Gauss' Divergence Theorem.**

(i) Gauss' Divergence Theorem: Let F be a vector field whose components have continuous first partial derivatives, and let S be a piecewise smooth oriented closed surface. The flux of F across S in the direction of the surface's outward unit normal field n equals the integral of the divergence $\nabla \cdot F$ over the region D enclosed by the surface, that is,

$$\iint_S F \cdot n \, d\sigma = \iiint_D \nabla \cdot F \, dV.$$

5. First order Ordinary Differential Equations